

W-ASSOCIAHEDRA ARE IN-YOUR-FACE

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ABSTRACT. We use a projection argument to uniformly prove that W -permutahedra and W -associahedra have the property that if v, v' are two vertices on the same face f , then any geodesic between v and v' does not leave f . In type A , we show that our geometric projection recovers a slight modification of the combinatorial projection in [STT88].

1. INTRODUCTION

The underlying graph of the Tamari lattice is the 1-skeleton of a classical associahedron associated to the Coxeter group A_n . This graph has vertices given by triangulations of a regular $(n + 3)$ -gon, with edges between two triangulations T, T' if one can obtain T' from T by flipping a single diagonal. In 1988, D. Sleator, R. Tarjan, and W. Thurston used a projection argument to prove that if T, T' share a common diagonal, then every shortest path between T and T' leaves this diagonal untouched [STT88].¹ Using type-dependent combinatorial models for the 1-skeleton of W -associahedra, this proof was recently extended by C. Ceballos and V. Pilaud to all classical types [CP14]. C. Ceballos and V. Pilaud were able to check many exceptional types by computer, but were unable to finish the verification for E_7 (4160 vertices) and E_8 (25080 vertices).

In this note, we supply a short uniform proof of this result, drawing heavily on the beautiful theory of N. Reading's Coxeter-sortable elements. We first review the in-your-face property in Section 2, summarizing the theory built in [STT88, CP14]. In Section 3, we then give a sesquipedalian proof that the W -permutahedron is in-your-face (Theorem 3.2). This provides the template for Section 4 and the main theorem of this paper, in which we prove that the W -associahedron is in-your-face (Theorem 4.7). The new idea we bring to this problem concerns how to define the projection—rather than work with type-specific combinatorial models, we define the projection geometrically. In Section 5, we show that our geometric projection recovers a slight modification of the combinatorial projection in [STT88].

Acknowledgments. I learned of this problem from T. Brüstle at the 2015 Commutative Algebra meets Algebraic Combinatorics conference at Queen's University in Kingston, Ontario, where he presented a proof his summer students had developed in type D_n . T. Brüstle has kindly informed me that he and J.-F. Marceau also have a uniform proof, using different methods [BM]. I would like to thank T. Brüstle, M. Guay-Paquet, A. Pang, and H. Thomas for useful conversations, C. Ceballos and V. Pilaud for their lucid presentation in [CP14], and C. Stump for helpful suggestions on presentation.

Date: February 23, 2015.

2000 Mathematics Subject Classification. Primary 05E45; Secondary 20F55, 13F60.

Key words and phrases. Coxeter–Catalan combinatorics, Coxeter-sortable elements, cluster complex, associahedron, permutahedron.

¹C. Ceballos and V. Pilaud call this the “non-leaving-face property” in [CP14]. We find our nomenclature a bit more on the nose.

2. THE IN-YOUR-FACE PROPERTY

Given a (finite) convex polytope \mathcal{P} with facets $\mathcal{F} = \{f\}_{f \in \mathcal{F}}$ and vertices $\mathcal{V} = \{v\}_{v \in \mathcal{V}}$, recall that its face poset is a finite, graded (by dimension), atomic and coatomic lattice. That is, an arbitrary face g may be uniquely specified by its set of containing facets

$$\mathcal{F}(g) = \{f \in \mathcal{F} : g \subseteq f\},$$

or by the set of vertices it contains

$$\mathcal{V}(g) = \{v \in \mathcal{V} : v \subseteq g\}.$$

For two vertices v, v' , we write $v \rightarrow v'$ if they are adjacent on the 1-skeleton of \mathcal{P} or if $v = v'$. A *path* from the vertex v to the vertex v' is a sequence

$$v = v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_k = v'.$$

A *geodesic* from vertices v to v' is a path of minimal length.

Definition 2.1 ([CP14, Definition 5]). A polytope \mathcal{P} is *in-your-face* if for all vertices $v, v' \in \mathcal{V}$, any geodesic from v to v' stays in the minimal face containing both.

We will not deviate from the original proof strategy of D. Sleator, R. Tarjan, and W. Thurston in [STT88], which was formalized by C. Ceballos and V. Pilaud [CP14]. Briefly, for each facet f , we define the notion of a “normalization map” $\phi_f : \mathcal{V} \rightarrow f$, and then show that the existence of such ϕ_f implies that \mathcal{P} is in-your-face.

Definition 2.2 ([STT88, Lemma 3], [CP14, Proposition 9]). A *normalization map* for the facet $f \in \mathcal{F}$ is a function $\phi_f : \mathcal{V} \rightarrow f$ such that:

- (1) $\phi_f(v) = v$ for $v \in \mathcal{V}(f)$;
- (2) if $v \rightarrow v'$, then $\phi_f(v) \rightarrow \phi_f(v')$; and
- (3) if $v \rightarrow v'$ with $v \in f$ but $v' \notin f$, then $\phi_f(v') = v$.

Lemma 2.3 ([STT88, Lemma 3], [CP14, Propositions 8 and 9]). *If a normalization map exists for each facet $f \in \mathcal{F}$, then \mathcal{P} is in-your-face.*

Proof. As in [STT88, CP14], we first show that if an element v is adjacent (but not on) the facet f containing v' , then we may find a geodesic between v and v' whose first step is onto the facet f .

To this end, let v, v', v'' be three vertices of \mathcal{P} such that v'' and v are adjacent, v'' and v' lie on the same face f , but v is not on the face f . We show that there is a geodesic between v and v' whose first step is $v \rightarrow v''$.

Let

$$v = v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_k = v'$$

be a geodesic between v and v' . Then the sequence (of length one greater than the original sequence)

$$v = v_0 \rightarrow \phi_f(v_0) \rightarrow \phi_f(v_1) \rightarrow \cdots \rightarrow \phi_f(v_k)$$

is actually a path between v and v' :

- by Property (1) of Definition 2.2, the last element is v_k ;
- by Property (2), the consecutive pairs besides the first are either adjacent or equal; and
- by Property (3), the first consecutive pair is adjacent, since we assumed that v is adjacent to a $v'' \in f$, so that $\phi_f(v_0) = v''$.

Since the first vertex v was not in the face f , but our final vertex v' was in f , we passed onto the face f at some step $v_i \rightarrow v_{i+1}$ in the original geodesic. Then $\phi_f(v_i) = \phi_f(v_{i+1})$, and we can remove this duplication to obtain a new geodesic whose first step is the edge $v \rightarrow v''$.

We now show that a geodesic between v and v' cannot leave a common facet. Suppose that v and v' lie on a common facet f , and suppose we have a geodesic

$$v = v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_k = v'$$

that leaves f . Let i be the minimal index so that the vertex v_i in the geodesic is not on f . By the previous argument, we have that the geodesic from v_i to v_k

$$v_i \rightarrow v_{i+1} \rightarrow \cdots \rightarrow v_k$$

may be normalized to a new geodesic from v_i to v_k

$$v_i = v'_i \rightarrow v'_{i+1} \rightarrow v'_{i+2} \rightarrow \cdots \rightarrow v'_k = v_k,$$

such that $v'_{i+1} = \phi_f(v_{i+1})$ lies on the face f . By assumption $v_{i-1} \rightarrow v_i$ left the face f , so by Property (2) we conclude that $\phi_f(v_{i+1}) = v_{i-1} = v'_{i+1}$.

Then we can merge the initial segment of the original geodesic from $v = v_0$ to v_{i-1} with the final segment of the new geodesic from v_{i-1} to $v' = v'_k$, so that

$$v = v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_{i-1} \rightarrow v'_{i+2} \rightarrow \cdots \rightarrow v'_k = v'$$

is a path from v to v' of length at least two shorter than the original supposed geodesic.

Finally, since this argument holds for any facet f containing both v and v' , it holds true for the minimal face containing both. \square

3. THE PERMUTAHEDRON

In this section, we use the framework of the previous section to prove that the W -permutahedron is in-your-face. Although this result can be proven much more quickly and intuitively, we feel justified in this long-winded approach because we will reuse the same ideas in Section 4 for the W -associahedron.

Given a *finite Coxeter system* (W, S) with $J \subseteq S$ a subset of the *simple reflections* S , the *standard parabolic subgroup* W_J is the Coxeter group generated by J , and the *parabolic quotient* is the weak order interval $\{w \in W : sw > w \text{ for all } s \in J\}$. If $S \setminus J = \{s\}$, we write $W_{\langle s \rangle}$ and $W^{(s)}$ for W_J and W^J . The *length* of $w \in W$ is the length $\ell(w)$ of a shortest expression for w as a product of simple reflections; we write $\text{Red}(w)$ for the set of all such shortest expressions, and call them *reduced words*. The *weak order* $\text{Weak}(W)$ is the lattice given by relations $u \leq v$ iff $\ell(u) + \ell(u^{-1}v) = \ell(v)$ [BB06, Theorem 3.2.1]. Recall that we may choose a system of *simple roots* $\Delta(W)$ which determine the *positive roots* $\Phi^+(W)$. We may order $\Phi^+(W)$ by $\alpha < \beta$ iff $\beta - \alpha$ is a nonnegative sum over elements of Δ . This order has a *highest root*. We associate to $w \in W$ its *inversion set* $\text{inv}(w) = (-w(\Phi^+(W))) \cap \Phi^+(W)$, and recall that W has a *longest element* w_o whose inversion set is all of $\Phi^+(W)$. We will use the notation $u^v := vuv^{-1}$, and we write $n := |S|$ and $N := |\Phi^+(W)|$.

All our examples will be in type A_3 ; in these examples, we label the simple reflections s_1, s_2 , and s_3 , so that $(s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_3)^2 = e$.

Proposition 3.1. *We recall the following facts with reference but without proof.*

- Every $w \in W$ has a unique factorization $w = w_J \cdot w^J$ with $w_J \in W_J$ and $w^J \in W^J$ [BB06, Proposition 2.4.4].
- The projection from $W \rightarrow W_J$ defined by $w \mapsto w_J$ is a lattice homomorphism from $\text{Weak}(W)$ to $\text{Weak}(W_J)$ [Rea04, Corollary 6.10].

We define the *W -permutahedron* $\text{Perm}(W)$ to be the convex hull of W -orbit of its highest root. It is well-known that the vertices of $\text{Perm}(W)$ are in bijection with elements $w \in W$, its faces are parametrized by the cosets W/W_J for $J \subseteq S$, and its facets are parametrized by the translations of maximal parabolic subgroups $\{wW_{\langle s \rangle} : s \in S, w \in W\}$.

An example is given in Figure 1.

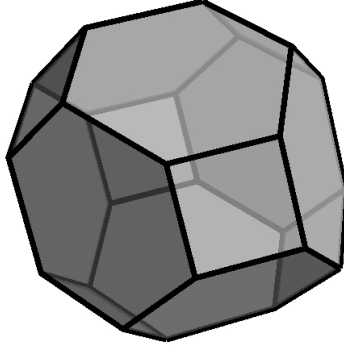


FIGURE 1. The permutahedron $\text{Perm}(A_3)$. The 24 elements of A_3 correspond to the 24 vertices. When correctly oriented, the 1-skeleton of $\text{Perm}(A_3)$ is isomorphic to $\text{Weak}(A_3)$.

Theorem 3.2. *The W -permutahedron is in-your-face.*

Proof. Suppose that w and w' lie in a common facet f . By multiplying W by w^{-1} , we orient the permutahedron so that this facet coincides with a maximal parabolic subgroup $W_{\langle s \rangle}$ of W . Since $u \mapsto w^{-1}u$ preserves all edges of the permutahedron, we are reduced to considering only facets of the form $f = W_{\langle s \rangle}$.

In this way, the normalization map naturally suggests itself: we define it as the projection

$$\mathfrak{P}_{W_{\langle s \rangle}} : W \rightarrow W_{\langle s \rangle}$$

by

$$\mathfrak{P}_{W_{\langle s \rangle}}(u) = u_{\langle s \rangle}.$$

By Proposition 3.1, $\mathfrak{P}_{W_{\langle s \rangle}}$ is clearly a map with image $W_{\langle s \rangle}$. We now check that $\mathfrak{P}_{W_{\langle s \rangle}}$ is a normalization map.

- (1) We immediately conclude that $\mathfrak{P}_{W_{\langle s \rangle}}(u) = u$ for $u \in W_{\langle s \rangle}$, since $u = u_{\langle s \rangle}$ when $u \in W_{\langle s \rangle}$.
- (2) We now check that if $u \rightarrow u'$, then $\mathfrak{P}_{W_{\langle s \rangle}}(u) \rightarrow \mathfrak{P}_{W_{\langle s \rangle}}(u')$. We may write the given edge $u \rightarrow u'$ as $u \rightarrow ut$, where $ut < u$ and $t \in S$. Write $u = u_{\langle s \rangle}u^{(s)}$; by the exchange lemma, t removes a reflection from either $u_{\langle s \rangle}$ or from $u^{(s)}$. In the former case, the edge is preserved: $u \rightarrow ut$ is sent to $u \rightarrow u_{\langle s \rangle}t'$, where $t' = t^{u^{(s)}}$. In the latter case, the edge is contracted.
- (3) Finally, we ensure that if $u \rightarrow u'$ with $u \in W_{\langle s \rangle}$ and $u' \notin W_{\langle s \rangle}$, then $\mathfrak{P}_{W_{\langle s \rangle}}(u') = u$. We write the given edge $u \rightarrow u'$ as $u \rightarrow us$, for $u \in W_{\langle s \rangle}$.

Since s does not appear in u , $u \cdot s$ is the parabolic decomposition of us with respect to $S \setminus \{s\}$, so that $(us)_{\langle s \rangle} = u$.

Since $\mathfrak{P}_{W_{\langle s \rangle}}$ is a normalization map, we conclude the Theorem by Lemma 2.3. \square

An example of this projection is given in Figure 2.

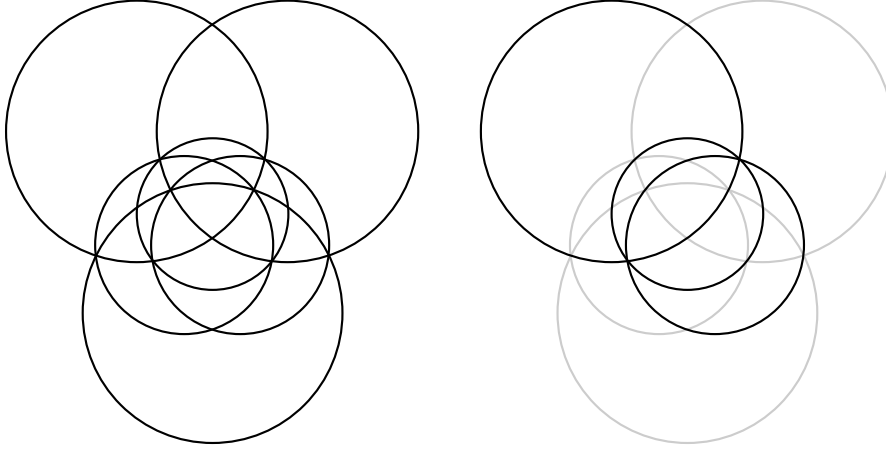


FIGURE 2. The picture on the left is the stereographic projection of the intersection of the hyperplane arrangement of type A_3 with a sphere (this is the dual of Figure 1). The six hyperplanes correspond to the six circles, and the 24 elements of A_3 correspond to the 24 connected regions—the identity element of A_3 corresponds to the central triangle, while the longest element corresponds to the unbounded region. The picture on the right is the restriction of A_3 to its maximal parabolic subgroup generated by the reflections s_2 and s_3 . The projection map sends an element w of A_3 to the unique element $w_{\langle s \rangle}$ of the parabolic subgroup $(A_3)_{\langle s_1 \rangle}$ that lies in the same connected region as w .

4. THE ASSOCIAHEDRON

In this section, we will show that the W -associahedron is in-your-face. The proof will follow exactly the same template as our proof for the W -permutahedron: we will describe facets, explain how to rotate these facets so that they lie in a parabolic subgroup of W , and then define the normalization map by projecting to this parabolic subgroup.

To make this program as smooth as possible, we require two different (W, c) -Catalan objects: subword complexes (on which it will be easy to describe the face structure of the W -associahedron, its 1-skeleton, and rotation), and sortable elements (on which it will be easy to understand parabolic subgroups and the normalization map).

4.1. Subwords and Facets of the Associahedron. Although there are many different realizations of W -associahedra, we only require a description of its faces and of its 1-skeleton. We therefore will not attempt to summarize the literature—for more information, the interested reader should consult [CSZ11, HLT11].

A simple description of the faces may be given using the work on subword complexes by J. P. Labbé, C. Ceballos, and C. Stump [CLS14], and by V. Pilaud and

C. Stump [PS11]. A (standard) *Coxeter element* $c = s_1 s_2 \cdots s_n$ is a product of the simple reflections, each occurring exactly once, in any order.

Fix a Coxeter element c and choose any reduced word $\mathbf{c} \in \text{Red}(c)$ (this choice is immaterial, since any two such words will agree up to commutations). The *c-sorting word* $\mathbf{w}(\mathbf{c}) \in \text{Red}(w)$ of w is the lexicographically first (in position) reduced S -subword for w of the word

$$\mathbf{c}^\infty = (s_1 s_2 \cdots s_n \mid s_1 s_2 \cdots s_n \mid s_1 s_2 \cdots s_n \mid \cdots).$$

For example, in type A_3 for $c = s_1 s_2 s_3$, $\mathbf{w}_o(\mathbf{c}) = s_1 s_2 s_3 | s_1 s_2 \cdot | s_1$.

Definition 4.1 ([CLS14]). Fix the word in simple reflections

$$\mathbf{Q} = (\mathbf{Q}_1, \mathbf{Q}_2, \dots, \mathbf{Q}_{N+n}) := \mathbf{c} \mathbf{w}_o(\mathbf{c}).$$

The (W, c) -subword complex $\text{Asso}(W, c)$ consists of all subwords of \mathbf{Q} whose complement contains a reduced word for w_o .

The vertices of the W -associahedron are in bijection with those subwords of $\text{Asso}(W, c)$ with n letters. Two vertices w and w' of $\text{Asso}(W, c)$ are connected by a *flip* if their corresponding subwords differ in a single position. The *flip graph* for $\text{Asso}(W, c)$ is the graph generated by flips, and it is isomorphic to the 1-skeleton of the W -associahedron [PS11]. An example of the flip graph for $\text{Asso}(A_3, s_1 s_2 s_3)$ is given in Figure 3.

Theorem 4.2 ([CLS14, PS11]). *In the language of $\text{Asso}(W, c)$, the faces of the W -associahedron are given by those subwords fixing a chosen collection of letters of \mathbf{Q} , and its facets are parametrized by single letters of \mathbf{Q} .*

It is not hard to see that the face structure of $\text{Asso}(W, c)$ does not depend on c (see Section 4.3), and we may therefore associate $\text{Asso}(W, c)$ with the *W -associahedron*.

4.2. Sortable Elements and Parabolic Subgroups. In order to describe the normalization map, we require the notion of a parabolic subgroup on (W, c) -Catalan objects. This is most easily introduced using N. Reading's c -sortable elements $\text{Sort}(W, c) \subseteq W$, which—as a subset of elements of W —inherit the parabolic structure of W .

Definition 4.3 ([Rea07b]). An element $w \in W$ is *c-sortable* if its c -sorting word defines a decreasing sequence of subsets of positions in \mathbf{c} . We denote the set of c -sortable elements by $\text{Sort}(W, c)$.

Let c' be the *restriction* of c to W_J , obtained by removing all simple reflections not in J from a reduced word \mathbf{c} . It is immediate from this definition that if w is c -sortable, then w_J is c' -sortable, and so we obtain the desired parabolic structure.

We now relate $\text{Sort}(W, c)$ and $\text{Asso}(W, c)$ to characterize the parabolic structure on $\text{Asso}(W, c)$. There is a natural bijection between the vertices of $\text{Asso}(W, c)$ and $\text{Sort}(W, c)$, which may be described using equivalent recursive structures on each set of objects, as in [PS11, Section 6.4.2] or [Rea07a]. In particular, an element $w \in \text{Sort}(W, c)$ is in W_J iff its corresponding subword (with n letters) does not use the simple reflections in J from the initial copy of \mathbf{c} in $\mathbf{Q} = \mathbf{c} \mathbf{w}_o(\mathbf{c})$. We conclude the following Proposition.

Proposition 4.4 ([CLS14, PS11]). *Let s be initial in \mathbf{c} and write $\mathbf{Q} = \mathbf{c} \mathbf{w}_o(\mathbf{c})$ so that its first letter \mathbf{Q}_1 is s . The facet $\{\mathbf{Q}_1\}$ of $\text{Asso}(W, c)$ contains the vertices recursively in bijection with those c -sortable elements in $W_{\langle s \rangle}$.*

In order to check that projection to the parabolic subgroup is a normalization map, we require reasonably explicit descriptions of the edges of the 1-skeleton of

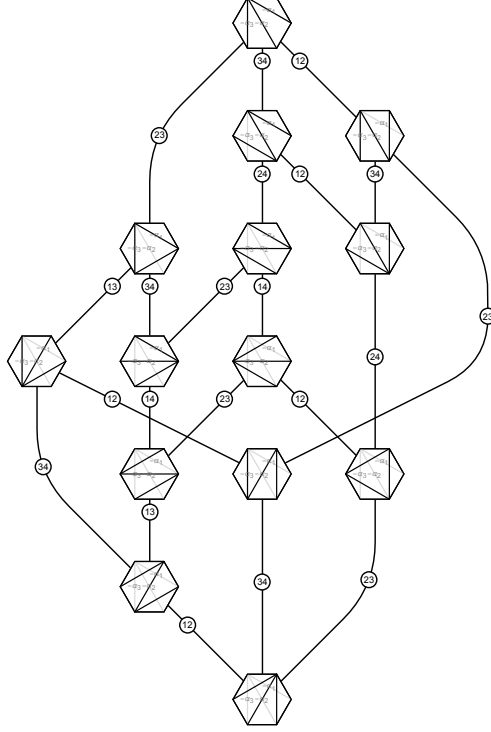


FIGURE 3. The flip graph for $\text{Asso}(A_3, s_1s_2s_3)$, where we have replaced subwords with their corresponding triangulations (a description of this bijection is given in [CLS14]; see also Section 5). The letters of \mathbf{Q} correspond to diagonals of the polygon, and moving a letter to flip from one vertex to another may be visualized as flipping the corresponding diagonal. When correctly oriented as a poset (as it is above), it is isomorphic to the Tamari lattice. We do not explain the edge labels here, but refer the reader to Figure 4.

the W -associahedron—and hence the edges of the flip graph of $\text{Asso}(W, c)$ —as c -sortable elements, and slightly more explicit descriptions of the edges incident to the parabolic subgroup $W_{\langle s \rangle}$.

Recall that W has the projection $\pi_{\downarrow}^c : W \rightarrow \text{Sort}(W, c)$, where

$$\pi_{\downarrow}^c(w) = \begin{cases} s\pi_{\downarrow}^{scs}(sw) & \text{if } w \geq s \\ \pi_{\downarrow}^{sc}(w_{\langle s \rangle}) & \text{if } w \not\geq s, \end{cases}$$

which by [RS11, Corollary 6.2] may be characterized as the unique maximal c -sortable element below w in the weak order. This projection respects the parabolic structure of W , in that it satisfies

$$\pi_{\downarrow}^c(w)_J = \pi_{\downarrow}^{c'}(w_J),$$

where c' is the restriction of c to W_J [RS11, Proposition 6.13].

Proposition 4.5. (1) *The underlying graph of the restriction of $\text{Weak}(W)$ to $\text{Sort}(W, c)$ is the 1-skeleton of the W -associahedron. Every edge may be*

expressed uniquely as $w \rightarrow \pi_{\downarrow}^c(ws)$ for some $w \in W$ and some $s \in S$ such that $ws < w$.

- (2) Let s be initial in c and let $w \in \text{Sort}(W_{\langle s \rangle}, sc)$. Then $v = s \vee w$ is the unique edge from w to an element not in $W_{\langle s \rangle}$.

Proof. The first statement follows [Rea06, Corollary 8.1].

For the second statement, we first note that since $v = s \vee w$ is the join of two c -sortable elements, it is itself c -sortable. Since a Cambrian lattice in rank n is n -regular, there is exactly one edge from w to an element outside of $W_{\langle s \rangle}$. We check that $w \rightarrow v$ is indeed this edge, which would follow if $\pi_{\downarrow}^c(vt) = w$ for some simple reflection t . Since s is a cover reflection of v by [Rea07b, Lemma 2.8], there is a simple reflection t so that $sv = vt$. Then $s \notin \text{inv}(vt)$, so that

$$\pi_{\downarrow}^c(vt) = \pi_{\downarrow}^{sc}((vt)_{\langle s \rangle}) = \pi_{\downarrow}^{sc}(w) = w,$$

since

$$\text{inv}((vt)_{\langle s \rangle}) = \text{inv}(vt) \cap \Phi^+(W_{\langle s \rangle}) = \text{inv}(v) \cap \Phi^+(W_{\langle s \rangle}) = \text{inv}(v_{\langle s \rangle}) = \text{inv}(w).$$

□

4.3. Orienting the Associahedron. To orient the W -associahedron—as we did with the W -permutahedron by multiplication—we require the notion of *Cambrian rotation*. Write $\bar{s} = s^{w_o} \in S$ and shift a subword in $\text{Asso}(W, c)$ one position to the left to obtain a subword of $Q' = s_{\pi_2} \cdots s_{\pi_n} w_o(c) \bar{s}$ (the leftmost letter s is sent to the rightmost letter \bar{s}). If we let $c' = s^{-1}cs$ with corresponding reduced word c' then, up to commutations, $s_{\pi_2} \cdots s_{\pi_n} w_o(c) \bar{s} = c' w_o(c')$, so that the resulting subword corresponds to a subword in $\text{Asso}(W, c')$ [CLS14]. This defines a map

$$\text{Camb}_s : \text{Asso}(W, c) \rightarrow \text{Asso}(W, s^{-1}cs),$$

and it is easy to see that this map does not change vertex adjacencies or the face structure. Using the characterization of facets from Proposition 4.4, the following Lemma follows immediately from the definition of Cambrian rotation.

Lemma 4.6. *Any facet f of $\text{Asso}(W, c)$ may be sent by Cambrian rotation to a facet f' of $\text{Asso}(W, c')$ for some c' , such that the vertices of f' correspond to sc' -sortable elements in $W_{\langle s \rangle}$, where s is initial in c' .*

4.4. The Main Theorem. We are now ready to prove the main theorem of this note.

Theorem 4.7. *The W -associahedron is in-your-face.*

Proof. Suppose that we have two vertices w and w' that lie in a common facet f of the W -associahedron. Thinking of w and w' as vertices of the subword complex, by Lemma 4.6 we can orient the associahedron so that f coincides with the set of sortable elements $\text{Sort}(W_{\langle s \rangle}, sc)$ in a maximal standard parabolic. Since Cambrian rotation preserves all edges of the W -associahedron, we are reduced to considering only facets of the form $f = \text{Sort}(W_{\langle s \rangle}, sc)$.

The normalization map again naturally suggests itself: we define

$$\mathfrak{C}_{W_{\langle s \rangle}} : \text{Sort}(W, c) \rightarrow \text{Sort}(W_{\langle s \rangle}, sc)$$

by

$$\mathfrak{C}_{W_{\langle s \rangle}}(u) = u_{\langle s \rangle}.$$

By the discussion directly after Definition 4.3, $\mathfrak{C}_{W_{\langle s \rangle}}$ is a map with image to $\text{Sort}(W_{\langle s \rangle}, sc)$. We now check that $\mathfrak{C}_{W_{\langle s \rangle}}$ is a normalization map.

- (1) We immediately conclude that $\mathfrak{C}_{W_{\langle s \rangle}}(u) = u$ for $u \in W_{\langle s \rangle}$, since $u = u_{\langle s \rangle}$ when $u \in W_{\langle s \rangle}$.

- (2) We now check that if $u \rightarrow u'$, then $\mathfrak{C}_{W_{\langle s \rangle}}(u) \rightarrow \mathfrak{C}_{W_{\langle s \rangle}}(u')$. By Proposition 4.5, we may write such an edge as $u \rightarrow \pi_{\downarrow}^c(ut)$, where $ut < u$ and $t \in S$. Now

$$\pi_{\downarrow}^c(ut)_{\langle s \rangle} = \pi_{\downarrow}^{sc}((ut)_{\langle s \rangle}),$$

by Proposition 6.13 in [RS11]. Using the same argument as in Theorem 3.2, we write $u = u_{\langle s \rangle} u^{\langle s \rangle}$ so that t removes a reflection from either $u_{\langle s \rangle}$ or from $u^{\langle s \rangle}$. In the former case, since $u_{\langle s \rangle}$ is c -sortable, the edge is preserved: $u \rightarrow \pi_{\downarrow}^c(ut)$ is sent to $u_{\langle s \rangle} \rightarrow \pi_{\downarrow}^c(u_{\langle s \rangle} t')$, where $t' = t^{u^{\langle s \rangle}}$. In the latter case, the edge is contracted.

- (3) Finally, we ensure that if $u \rightarrow u'$ with $u \in W_{\langle s \rangle}$ and $u' \notin W_{\langle s \rangle}$, then $\mathfrak{C}_{W_{\langle s \rangle}}(u') = u$. By Proposition 4.5, we may write such an edge as $u \rightarrow u'$, where $u' = s \vee u$. Now $u' = s \vee u'_{\langle s \rangle}$ [Rea07b, Lemma 2.7], so that because

- $u'_{\langle s \rangle}$ is sc -sortable in $W_{\langle s \rangle}$,
- c -sortable elements are uniquely characterized by their cover reflections [Rea07b, Proposition 2.5], and
- $\text{cov}(u') = \text{cov}(s \vee u) = \text{cov}(u) \cup \{s\}$ [Rea07b, Lemma 2.8],

we conclude that

$$u = u'_{\langle s \rangle} = \mathfrak{C}_{W_{\langle s \rangle}}(u').$$

Since \mathfrak{C}_f is a normalization map, we conclude the Theorem by Lemma 2.3. \square

An example of this projection is given in Figures 4 and 5.

5. COMBINATORIAL MODELS

In this section, we review the combinatorics of type A , and prove that our projection recovers a slight modification of the projection defined in [STT88].

5.1. Triangulations. We first recall the combinatorial model for triangulations in type A_{n-1} for a standard Coxeter element c [Rea06, CLS14]. The element c is an n -cycle of the form

$$c := (1, d_1, d_2, \dots, d_l, n, u_k, u_{k-1}, \dots, u_1)$$

so that the *lower numbers* d_i satisfy $d_1 < d_2 < \dots < d_l$ and the *upper numbers* u_i satisfy $u_1 < u_2 < \dots < u_k$.

We use the n -cycle above to describe an unconventional method for labeling an $(n+2)$ -gon. We will first describe the vertices of this $(n+2)$ -gon. We fix a circle of radius $(n+1)$ with left-most point at $(0,0)$. We will refer to the point above the x -axis with x -coordinate equal to i as the *upper vertex* i , and similarly call the corresponding point below the x -axis the *lower vertex* i . We mark the vertices $(0,0)$ and $(n+1,0)$, as well as all upper vertices d_i and all lower vertices u_i . We may freely choose to mark either the upper or the lower vertex of the circle for the x -coordinates 1 and n , so that there is exactly one marked vertex on the circle for each x -coordinate from 0 to $n+1$.

We now recall N. Reading's elegant bijection between c -sortable elements and triangulations of this $(n+2)$ -gon [Rea06]. Fix a c -sortable element w , let λ_0 be the path from $(0,0)$ to $(n+1,0)$ that passes along the lower part of the circle, and let $w = w_1 w_2 \dots w_n$ be the one-line notation for w . We now read the one-line notation for w from left to right. At the k -th step, we have a previously-constructed path P_{k-1} , and we have reached the k th letter w_k of w 's one-line notation. If w_k is a lower number, then P_k is equal to P_{k-1} without the lower vertex w_k ; otherwise, if w_k is an upper number, then P_k is equal to P_{k-1} with the upper vertex w_k added. Then the union of the paths P_k for $1 \leq k \leq n$ is the drawing of a triangulation of

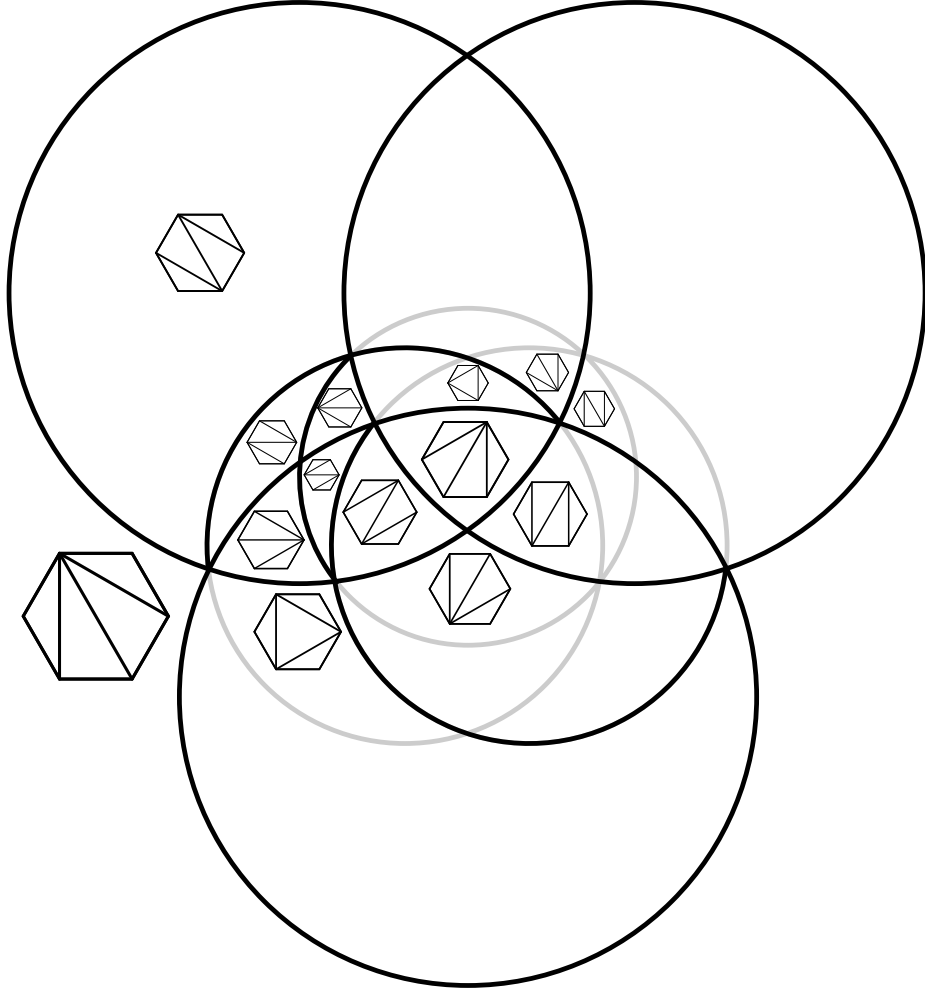


FIGURE 4. The stereographic projection of the intersection of the hyperplane arrangement of type A_3 with a sphere, where the arcs in black are those edges of the permutahedron preserved under the Cambrian congruence for $c = s_1 s_2 s_3$. The connected regions (using only the arcs in black) are indexed by vertices of the subword complex (which we draw as triangulations, as in Figure 3). The minimal connected regions (when using both gray and black arcs) in the connected regions (using only black arcs) are the c -sortable elements.

the un conventionally-labeled $(n+2)$ -gon. We shall denote this triangulation $T_c(w)$. An example is given in Figure 6.

We refer the reader to Table 1 of [CLS14] for the corresponding map from subwords to triangulations, whereby a unique diagonal of an $(n+2)$ -gon is associated to each facet of the associahedron (by Theorem 4.4, this amounts to associating a diagonal to each letter of the word $Q = cw_o(c)$). The important point we require is that—up to a global conventional relabeling of the un conventionally-labeled $(n+2)$ -gon—a c -sortable element w and its Cambrian rotation are both assigned the same triangulation (note that the description in [CLS14] has already imposed this global

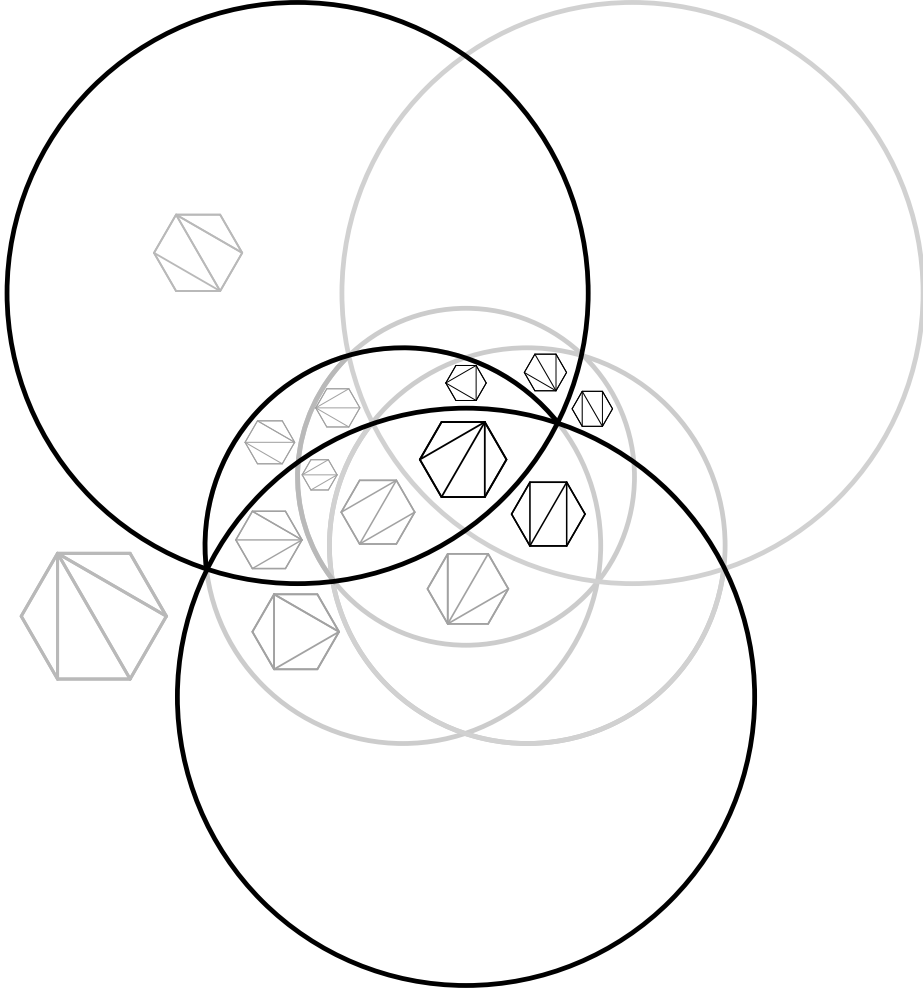


FIGURE 5. The restriction of Figure 4 to the maximal parabolic subgroup generated by s_2 and s_3 . The projection map sends an element w of $\text{Sort}(A_3, c)$ to the unique element $w_{\langle s \rangle}$ of $\text{Sort}((A_3)_{\langle s_1 \rangle}, s_2 s_3)$ that lies in the same connected region as w . It is geometrically clear that this projection either preserves or contracts edges.

relabeling). As a particular case, applying Cambrian rotation in the order given by a fixed reduced word for c gives a map from $\text{Asso}(A_{n-1}, c)$ to $\text{Asso}(A_{n-1}, c)$ that corresponds to simply *rotating* the triangulation.

5.2. A Combinatorial Projection. In this section, we give a slightly modified version of D. Sleator, R. Tarjan, and W. Thurston's type A_n combinatorial projection.

We will label the simple reflections so that s_i has cycle notation $(i, i+1)$. Let

$$c_i = s_i s_{i+1} \cdots s_{n-1} s_1 s_2 \cdots s_{i-1} = (1, 2, \dots, i-1, i+1, \dots, n, i),$$

so that we may take the single upper vertex of our unconventionally-labeled $(n+2)$ -gon corresponding to c_i to be the upper vertex i .

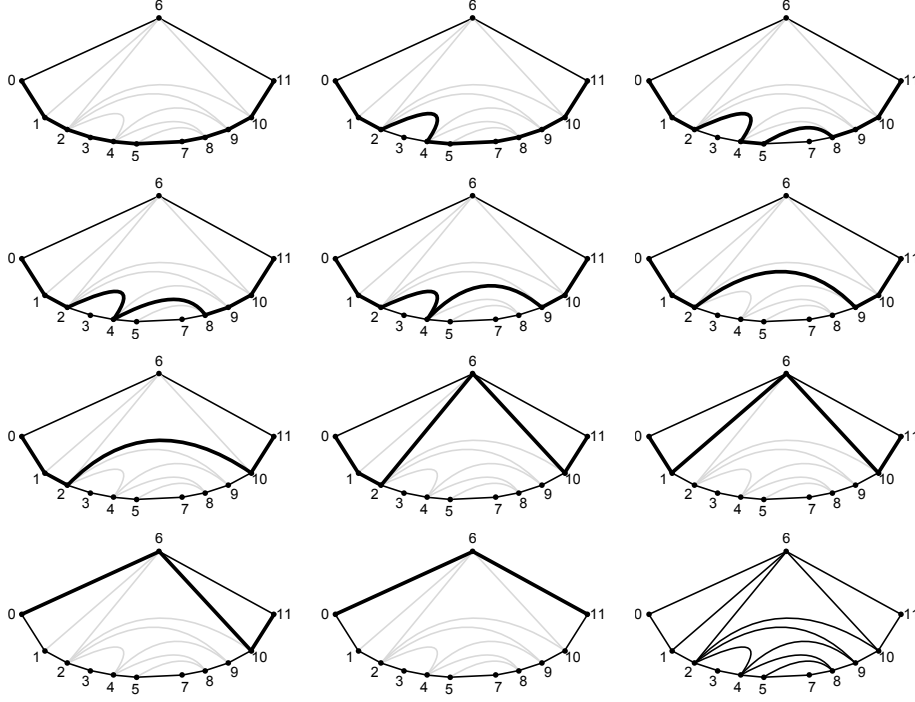


FIGURE 6. An illustration of the bijection T_c between an element w and $T_c(w)$, for $c = s_6 s_7 s_8 s_9 s_1 s_2 s_3 s_4 s_5$ and w with one-line notation $3, 7, 5, 8, 4, 9, 6, 2, 1, 10$.

Given a $(n+2)$ -gon with vertices labeled counterclockwise in the usual way by $0, 1, 2, \dots, n+1$, and given a diagonal $f = (f_1, f_2)$ with $f_1 < f_2$, there are exactly two ways relabel the vertices of this $(n+2)$ -gon so that the labeling corresponds to some c_i and so that the new labels of the diagonal f are i and $i+1$. (One may take either the vertex directly counterclockwise of f_1 or directly counterclockwise of f_2 to be the vertex whose new label is 0, and from there label vertices counterclockwise by $1, 2, \dots, i-1, i+1, i+2, \dots, n+1, i$). Choose one—this is the choice to project to either the parabolic subgroup $A_i \times A_{n-i-1}$ or to the parabolic subgroup $A_{n-i-1} \times A_i$. That we have a choice at all is, as usual, an artifact of the cute but annoying fact that $w_o s_i w_o = s_{n-i}$ in type A_{n-1} . We note that this choice of relabeling is irrelevant: for either choice, our projection map will define the same triangulation up to rotation.

We now give the combinatorial projection \mathfrak{C}_i . Fix the diagonal $(i, i+1)$ of the labeled $(n+2)$ -gon arising from c_i , and define D_i to be the set of interior diagonals, specified as pairs of vertices. Given a diagonal (a, b) with $a < b$, we set

$$\mathfrak{C}_i((a, b)) := \begin{cases} \{(a, i+1), (i, b)\} \cap D_i & \text{if } a < i \text{ and } b > i \\ \{(a, b)\} & \text{otherwise.} \end{cases}$$

Given a triangulation $T = \{(a_p, b_p) : 1 \leq p \leq n-1\}$, we define

$$\mathfrak{C}_i(T) := \{(i, i+1)\} \cup \bigcup_p \mathfrak{C}_i((a_p, b_p)).$$

It is easy to check, using the same reasoning as in [CLS14], that \mathfrak{C}_i is a normalization map. An example is given by the triangulations in Figure 7.

Remark 5.1. We stress here that this differs from the projection in [STT88, CLS14], which in our setup would be given by

$$N_{(i,i+1)}((a,b)) := \begin{cases} \{(a,x), (x,b)\} \cap D_i & \text{if } a < i \text{ and } b > i \\ \{(a,b)\} & \text{otherwise,} \end{cases}$$

where x is chosen beforehand to (always) be either i or $i+1$. Our projection has the benefit of eliminating this choice.

5.3. Equivalence of Projections. Using the bijection T_{c_i} , we will now show that our modification \mathfrak{C}_i of D. Sleator, R. Tarjan, and W. Thurston's combinatorial projection recovers the geometric projection $\mathfrak{C}_{W_{(i)}}$ defined in Section 4.4.

We fix c to be the n -cycle $c_1 = s_1 s_2 \cdots s_{n-1} = (1, 2, \dots, n)$. By the discussion in Section 4.3, any facet of $\text{Asso}(A_{n-1}, c)$ may be rotated to become the c_i -sortable elements in the standard parabolic subgroup $W_{(s_i)}$, for some i . This face is specified by the diagonal $(i, i+1)$.

Theorem 5.2. *Let w be a c_i -sortable element. Then*

$$T_{c_i}(\mathfrak{C}_{W_{(s_i)}}(w)) = \mathfrak{C}_i(T_{c_i}(w)).$$

An example is given in Figure 7.

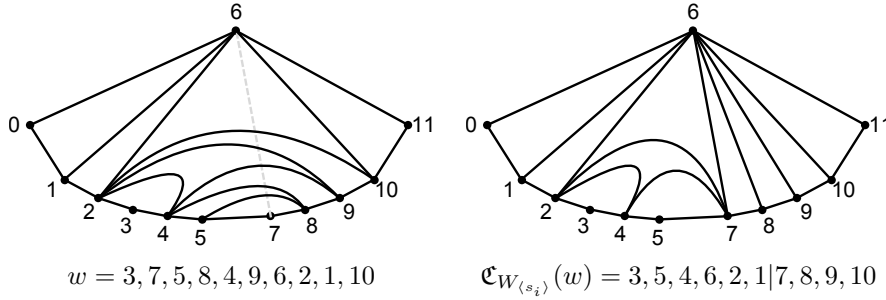


FIGURE 7. On the left is a permutation w below its corresponding triangulation. On the right is the permutation $\mathfrak{C}_{W_{(s_i)}}(w)$ below its triangulation, illustrating Theorem 5.2

Proof. We first establish the necessary properties of one-line notation, to better understand the map from sortable elements to triangulations.

It is easy to see that the parabolic subgroup $W_{(s_i)}$ consists of those permutations whose one-line notation has the property that each letter $1, 2, \dots, i$ is to the left of all letters $i+1, i+2, \dots, n$. Then by the bijection between $c_{(i)}$ -sortables and triangulations above, the $c_{(i)}$ -sortable elements that lie in $W_{(s_i)}$ correspond to exactly those triangulations that have the diagonal $(i, i+1)$.

Let $w = w_1 w_2 \cdots w_n$ be the one-line notation of w , let $w_{j_1} w_{j_2} \cdots w_{j_i}$ be its (in-order) restriction to those $w_j \leq i$, and let $w_{j_{i+1}} w_{j_{i+2}} \cdots w_{j_n}$ be its restriction to those $w_k > i$. Recall that

$$\text{inv}(\mathfrak{C}_{W_{(s_i)}}(w)) = \text{inv}(w) \cap \text{inv}((w_o)_{(s_i)}).$$

Then $u = \mathfrak{C}_{W_{(s_i)}}(w)$ has one-line notation $u_1 u_2 \cdots u_n$, where the first i letters are the numbers from 1 to i and their relative ordering in w is preserved, and similarly for the last $n+1-i$ letters. Precisely,

$$u_1 u_2 \cdots u_i = w_{j_1} w_{j_2} \cdots w_{j_i} \text{ and } u_{i+1} u_{i+2} \cdots u_n = w_{j_{i+1}} w_{j_{i+2}} \cdots w_{j_n}.$$

The theorem now follows immediately by noting that the lower vertex $i + 1$ is contained in each of the paths P_k for $1 \leq k \leq i$, while the upper vertex i is contained in each of the paths P_k for $i + 1 \leq k \leq n$. Furthermore, the above considerations make it clear that this equality actually holds for any element of A_{n-1} . \square

6. FINAL REMARKS

It would be interesting to compare the combinatorial projection maps defined for types B and D in [CP14] with the combinatorics one could extract from our geometric projection. One would also expect that such a result holds for the Fuss-analogues of associahedra (despite the absence of hyperplane geometry in this situation, the projection map would be defined by projecting onto what passes for a standard parabolic in the positive Artin monoid). Finally, we note that the similarity of the proofs for the W -permutahedron and the W -associahedron suggest that the proper generality for an in-your-face theory has not been found.

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